Bridge Cohomology

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For any $\Bbbk$-algebra A we have the Hochschild and Bar cochain complexes $C^\bullet_{\mathrm{bar}}(A)$ and $C^\bullet(A)$, where $C^n_{(\mathrm{bar})}(A)$ is the module of $n+1$ multilinear functionals on A, and the boundary maps are respectively given by

$$b'\varphi(a_0, \ldots, a_{n+1}) = \sum_{i=0}^{n} (-1)^i \varphi(a_0, \ldots, a_i a_{i+1}, \ldots, a_{n+1})$$

and

$$b\varphi(a_0, \ldots, a_{n+1}) = b'\varphi(a_0, \ldots, a_{n+1}) + (-1)^{n+1}\varphi(a_{n+1}a_0, \ldots, a_n)$$

For any algebra A (not necessarily unital) the bar cohomology of A is the cohomology of the complex $C^\bullet_{\mathrm{bar}}(A)$

$$HB^\bullet(A) := H^\bullet(C^\bullet_{\mathrm{bar}}(A))$$

When A is a unital algebra, the Hochschild cohomology of A is defined as the cohomology of the complex $C^\bullet(A)$

$$HH^\bullet(A) := H^\bullet(C^\bullet(A))$$

The *Reduced Hochschild cochain complex* is composed of the modules

$$C^n_{\mathrm{red}}(A) = \{\varphi \,|\, \varphi(a_0, ..., a_n) = 0 \text{ if } a_i = 1,\ 1 \leq i \leq n\}$$

for $n \geq 1$, and

$$C^0_{\mathrm{red}}(A) = \{\varphi \,|\, \varphi(1) = 0\}$$

The *reduced Hochschild cohomology* is then

$$\overline{HH}^\bullet(A) := H^\bullet(C^\bullet(A)_{\mathrm{red}})$$

For a non-unital algebra A, $HH^n(A) = H^n(C^\bullet(A_+)_{\mathrm{red}})$.

There exist chain maps

$$(1-\lambda) : C^\bullet(A) \to C^\bullet_{\mathrm{bar}}(A)$$

$$Q : C^\bullet_{\mathrm{bar}}(A) \to C^\bullet(A)$$

Where

$$\lambda\varphi(a_0, \ldots, a_n) = (-1)^n \varphi(a_n, a_0, \ldots, a_{n-1})$$

and

$$Q = \sum_{i=0}^{n} \lambda^i$$

When $\Bbbk$ contains $\mathbb{Q}$ the sequence is exact

$$\ldots \xrightarrow{Q} C(A) \xrightarrow{1-\lambda} C_{\mathrm{bar}}(A) \xrightarrow{Q} C(A) \xrightarrow{1-\lambda} C_{\mathrm{bar}}(A) \xrightarrow{Q} \ldots$$

Definition

The Connes complex $C_\lambda(A)$ is given as the kernel of $1 - \lambda$:

$$0 \to C_\lambda(A) \to C(A) \xrightarrow{1-\lambda} C_{\mathrm{bar}}(A)$$

The "Cyclic" Cohomology of A is then $H^\bullet_\lambda(A) := H^\bullet(C_\lambda(A))$.

Cyclic Cohomology

Definition

For any algebra A the *cyclic cobicomplex* $CC^{\bullet\bullet}(A)$ is the bicomplex

$$
\begin{array}{ccccccccc}
b\big\uparrow & & -b'\big\uparrow & & b\big\uparrow & & -b'\big\uparrow & & \\
C^2(A) & \xrightarrow{1-\lambda} & C^2_{\mathrm{bar}}(A) & \xrightarrow{Q} & C^2(A) & \xrightarrow{1-\lambda} & C^2_{\mathrm{bar}}(A) & \xrightarrow{Q} & \\
b\big\uparrow & & -b'\big\uparrow & & b\big\uparrow & & -b'\big\uparrow & & \\
C^1(A) & \xrightarrow{1-\lambda} & C^1_{\mathrm{bar}}(A) & \xrightarrow{Q} & C^1(A) & \xrightarrow{1-\lambda} & C^1_{\mathrm{bar}}(A) & \xrightarrow{Q} & \\
b\big\uparrow & & -b'\big\uparrow & & b\big\uparrow & & -b'\big\uparrow & & \\
C^0(A) & \xrightarrow{1-\lambda} & C^0_{\mathrm{bar}}(A) & \xrightarrow{Q} & C^0(A) & \xrightarrow{1-\lambda} & C^0_{\mathrm{bar}}(A) & \xrightarrow{Q} &
\end{array}
$$

The n^{th} *cyclic cohomology of A* is then

$$HC^n(A) := H^n(\mathrm{Tot}\, CC^{\bullet\bullet}(A))$$

Theorems and Examples

Proposition (Connes)

Let τ be an $n+1$ linear functional on A. Then the following are equivalent:

1. *There is an n-dimensional cycle $(\Omega, d, \int)$ and a homomorphism $\rho : A \to \Omega^0$ such that*

$$\tau(a_0, ..., a_n) = \int \rho(a_0)d\rho(a_1)...d\rho(a_n)$$

2. *There exists a closed graded trace $\hat{\tau}$ of dimension n on $\Omega^*(A)$ such that*

$$\tau(a_0, ..., a_n) = \hat{\tau}(a_0da_1...da_n)$$

3. $b\tau = 0$ *and* $(1-\lambda)\tau = 0$. *That is* $\tau \in Z^n_\lambda(A)$.

Theorems and Examples

There exist pairings $\langle K_0(A), HC^e(A)\rangle$ and $\langle K_1(A), HC^o(A)\rangle$ between the first and second K-theory groups of A and the even and odd cyclic cohomological groups of A.

An open question is then, how can we apply these results to manifolds with boundary?

Theorems and Examples

Specifically, if we look at the proof of the previous proposition 1) $\implies$ 3):

$$\tau(a_0, ..., a_n) = \int a_0 da_1 ... da_n$$

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$$\begin{aligned}\tau(a_0, ..., a_n) &= \int a_0 da_1 ... da_n \\ &= (-1)^{n-1} \int da_n a_0 da_1 ... da_{n-1} \\ &= (-1)^{n-1} \int d(a_n a_0) da_1 ... da_{n-1} + (-1)^n \int a_n da_0 da_1 ... da_{n-1}\end{aligned}$$

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&= (-1)^{n-1} \int d(a_n a_0) da_1 ... da_{n-1} + \lambda \tau(a_0, ..., a_n)
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&= (-1)^{n-1} \int d(a_n a_0) da_1 ... da_{n-1} + \lambda \tau(a_0, ..., a_n)
\end{aligned}$$

Hence

$$(1 - \lambda)\tau(a_0, ..., a_n) = (-1)^{n-1} \int d(a_n a_0) da_1 ... da_{n-1} \sim \int_{\partial M} \alpha$$

Theorems and Examples

For a manifold M with Boundary ∂M, we now have two algebras $A = C^\infty(M)$ and $B = C^\infty(\partial M)$ (or $\mathscr{E}^\infty(\partial M)$) along with a surjection $A \xrightarrow{\sigma} B$ between them, and we are looking for functionals $\varphi \in C^\bullet(A)$ such that $(1-\lambda)\varphi \in \sigma^* C^\bullet(B)$

Bridge Cohomology

(Originally "Restricted Cyclic Cohomology")

Definition

For a surjective map of unital algebras $A \xrightarrow{\sigma} B$, the bridge complex $R(\sigma)$ can be defined as the pullback in the following diagram:

$$\begin{array}{ccc} R^\bullet(\sigma) & \longrightarrow & C^\bullet(A) \\ \downarrow & \ulcorner & \downarrow {\scriptstyle 1-\lambda} \\ C^\bullet_{\mathrm{bar}}(B) & \xrightarrow{\sigma^*} & C^\bullet_{\mathrm{bar}}(A) \end{array}$$

$$R^n(\sigma) = \left\{ \begin{pmatrix} \varphi \\ \psi \end{pmatrix} \in C^n(A) \times C^n_{\mathrm{bar}}(B) \,\middle|\, (1-\lambda)\varphi = \sigma^*\psi \right\}, \begin{pmatrix} b & 0 \\ 0 & b' \end{pmatrix}$$

Bridge Cohomology

Of special note:

$$\begin{aligned} R^\bullet(\mathrm{id}_A) &= \{\varphi \in C^\bullet(A) \,|\, (1-\lambda)\varphi = \mathrm{id}_A^*\psi \text{ for some } \psi \in C^\bullet(A)\} \\ &= C^\bullet(A) \end{aligned}$$

And for the zero map we have the short exact sequence $A \to A \xrightarrow{0_A} 0$, and bridge cohomology

$$R^\bullet(0_A) = \{\varphi \in C^\bullet(A) \,|\, (1-\lambda)\varphi = 0\} = C^\bullet_\lambda(A).$$

Definition

Given any $\Bbbk$-algebras A and B (not necessarily unital) and a surjective algebra homomorphism $\sigma : A \to B$, let $\sigma_+ : A_+ \to B_+$. We define the n^{th} bridge cohomology module of σ as:

$$HR^n(\sigma) := \ker\left(HR^n(\sigma_+) \xrightarrow{\iota^*} HR^n(\mathrm{id}_{\Bbbk})\right).$$

Bridge Cohomology

Normalized and Reduced Complexes

Proposition-Definition

$R(\sigma)_{\mathrm{red}}$ is the pullback of the corresponding normalized complexes:

$$\begin{array}{ccc} R^\bullet(\sigma)_{\mathrm{red}} & \longrightarrow & C^\bullet(A)_{\mathrm{red}} \\ \downarrow & & \downarrow{\scriptstyle 1-\lambda} \\ C^\bullet_{\mathrm{bar}}(B)_{\mathrm{red}} & \xrightarrow{\sigma^*} & C^\bullet_{\mathrm{bar}}(A)_{\mathrm{red}} \end{array}$$

Theorem

For a non-unital surjection $\sigma : A \to B$, $HR^n(\sigma) = H^n(R(\sigma_+)_{\mathrm{red}})$.

Let $\sigma : A \to A/I$ be a surjective unital algebra homomorphism, where $I \subset A$ is an ideal. Then the relative Hochschild complex $C(A, I)$ is defined as the cokernel

$$0 \to C(A/I) \to C(A) \to C(A, I) \to 0.$$

With cohomology $HH(A, I)$.

Definition

Let A be a unital algebra and $0 \to I \to A \xrightarrow{\sigma} A/I \to 0$ be a short exact sequence of algebras. Define the *bridge bicomplex of* σ, $RR(\sigma)$, as the bicomplex with the following columns

$$C(A) \xrightarrow{q(1-\lambda)} C_{\mathrm{bar}}(A,I) \xrightarrow{Q} C(A,I) \xrightarrow{1-\lambda} C_{\mathrm{bar}}(A,I) \xrightarrow{Q} \dots$$

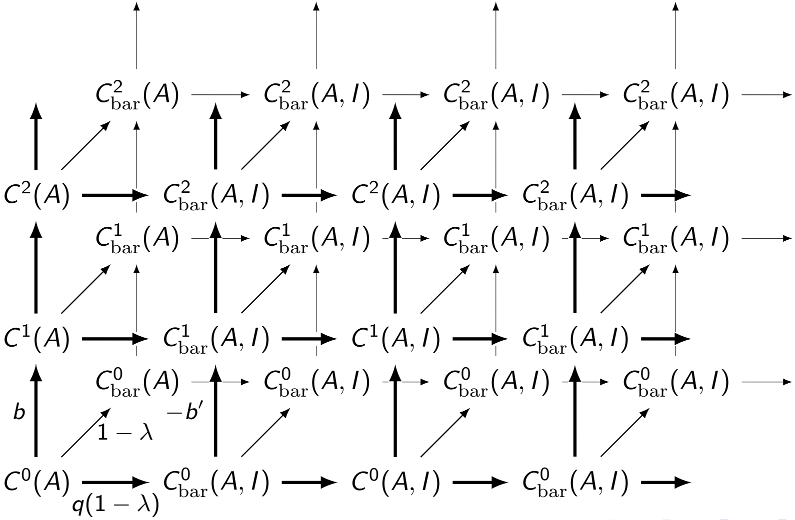
With cohomology given by $HR^n(\sigma) := H^n(\mathrm{Tot}\, RR(\sigma))$ called the *bridge cohomology of* σ.

Proposition

When $\Bbbk$ contains $\mathbb{Q}$, the total complex of $RR(\sigma)$ is quasi-isomorphic to the bridge complex, $\mathrm{Tot}\, RR(\sigma) \overset{q}{\cong} R(\sigma)$.

Lemma

For an augmented morphism $\sigma_+ : A_+ \to A_+/I$,
$HR^n(\sigma) = H^n(\operatorname{Tot} RRB(\sigma))$, *where* $RRB(\sigma)$ *is the following tricomplex*



Theorem (B.,Lesch, Moscovici, Pflaum)

There exists cohomological long exact sequences

$$\cdots \to HC^n(A,I) \xrightarrow{\widetilde{S}} HR^{n+2}(\sigma) \xrightarrow{I} HH^{n+2}(A) \xrightarrow{\widetilde{B}} HC^{n+1}(A,I) \to \dots$$

and

$$\cdots \to H^n_\lambda(A) \xrightarrow{I} HR^n(\sigma) \xrightarrow{B} H^{n-1}_\lambda(A/I) \xrightarrow{\sigma^*\circ S} H^{n+1}_\lambda(A/I) \to \dots$$

Given an exact sequence

$$0 \to \sigma \xrightarrow{(f_1,f_2)} \tau \xrightarrow{(g_1,g_2)} \tau/\sigma \to 0$$

we can define the relative bridge cocomplex, $R^\bullet(\tau,\sigma)$, as the cokernel

$$0 \to R^\bullet(\tau/\sigma) \to R^\bullet(\tau) \to R^\bullet(\tau,\sigma) \to 0$$

Theorem (B.)

(Excision) Let $0 \to \rho \to \sigma \to \tau \to 0$ be a copure short exact sequence in the category $\mathcal{S}_{\Bbbk}$, with associated nine diagram

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & I \cap K & \longrightarrow & K & \xrightarrow[\rho]{} & K/I \cap K & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & I & \longrightarrow & A & \xrightarrow{\sigma} & A/I & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow{\scriptstyle q_1} & & \downarrow{\scriptstyle q_2} & & \\
0 & \longrightarrow & I/I \cap K & \longrightarrow & A/K & \xrightarrow{\tau} & A/(I+K) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

Then the map $R(\sigma, \rho) \to R(\rho)$ is a quasi-isomorphism if and only if K and $K/K \cap I$ are coH-unital.

Research Goal

Correlate bridge cohomology and de Rham Homology on manifolds with boundary: (L.,M.,P.) for M compact and

$$\mathscr{J}^{\infty}(\partial M; M) \to C^{\infty}(M) \xrightarrow{\sigma} \mathscr{E}^{\infty}(\partial M),$$

$$HR^{k}(\sigma) \cong B^{-1}(\mathscr{D}'_{k-1}(M;\partial M)) \oplus H^{dR}_{k-2}(M;\partial M) \oplus H^{dR}_{k-4}(M;\partial M) \oplus \ldots$$

Extend the pairings $\langle K_0(A), HC^{e}(A)\rangle$ *and* $\langle K_1(A), HC^{o}(A)\rangle$ *from Connes, to manifolds with boundaries.*

Thank You!